

Periodic first integrals for Hamiltonian systems of Lie type.

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Abstract

We prove the existence of a Lie algebra of first integrals for time dependent Hamiltonian systems of Lie type. Moreover, applying the Floquet theory for periodic Euler systems on Lie algebras, we show the existence of an abelian Lie algebra of periodic first integrals for periodic Hamiltonian systems. An application to the dynamics of a nonlinear oscillator is given.

1 Introduction.

The existence of first integrals for time dependent Hamiltonian systems is a very important topic in the theory of differential equations and its applications. In general, for such class of systems, there is no time independent constants of motion. This is true even for 1-dimensional classical Hamiltonian systems of the form

$$H(q, p, t) = \frac{1}{2}p^2 + V(t, q)$$

corresponding to the motion of a particle under a time dependent potential $V(t, q)$.

In this article, we deal with the class of time dependent Hamiltonian systems which can be represented as a linear combination of Hamiltonian systems closing under the Lie bracket in a finite dimensional Lie algebra and having as coefficients scalar functions of time. These systems has been studied by S. Lie and can be considered as a generalization of linear systems but with a nonlinear superposition rule [1]. Recently the study of Lie systems has revived and new approaches and applications has been done by Ibrahimov [2], Winternitz and coworkers [3] and Cariñena and coworkers [4][5].

In this article, we study the existence of first integrals for Lie systems generated by Hamiltonian vector fields closing in a finite Lie algebra. Under the assumption of the existence of Hamiltonian functions closing under the Poisson bracket in a Lie algebra isomorphic to that generated by the original vector

fields, we prove the existence of a Lie algebra of first integrals for time dependent Hamiltonian systems of Lie type. Moreover, applying the Floquet theory for periodic Euler systems on Lie algebras, we prove for T -periodic Hamiltonian systems of Lie type, the existence of an abelian Poisson algebra of $2T$ -periodic first integrals and we also give conditions for the existence of a Lie algebra of T -periodic first integrals.

Finally, we include an application to the Milne-Pinney system that describes the time evolution of an isotonic oscillator [8], [9] founding for that case a periodic first integral similar to that given by Lewis for time depending oscillators [12].

2 Time dependent Hamiltonian systems of Lie type.

Let be (P, ω) a symplectic manifold. A time dependent Hamiltonian vector field X on P is called a *Hamiltonian vector field of Lie type* if X can be written in the form

$$X(t, x) = \sum_{i=1}^n b_i(t) X_i(x), \quad t \in \mathbb{R}, \quad x \in M \quad (1)$$

where $\{b_i(t) \mid i = 1, \dots, n\}$ are smooth real functions and the vector fields $\{X_i \mid i = 1, \dots, n\}$ are Hamiltonian vector fields which close under the Lie bracket on a n -dimensional real Lie algebra \mathfrak{g} of vector fields, i.e. there exists n^3 real numbers λ_{ij}^k such that

$$[X_i, X_j] = \sum_{k=1}^n \lambda_{ij}^k X_k, \quad \forall i, j = 1, \dots, n.$$

If the scalar functions $b_i(t)$ are T -periodic, $b_i(t+T) = b_i(t)$, $i = 1, \dots, n$, the Lie system is called a *T -periodic Hamiltonian system of Lie type*.

Let us take Hamiltonian functions $H_i(x)$, $i = 1, \dots, n$ associated to the basis for \mathfrak{g} given by the vector fields X_i , $i = 1, \dots, n$, and suppose we can choose Hamiltonian functions H_X for the Hamiltonian vector fields X of \mathfrak{g} in such way that

$$H_{\alpha X + \beta Y} = \alpha H_X + \beta H_Y, \quad \alpha, \beta \in \mathbb{R} \quad (2)$$

$$\{H_X, H_Y\} = H_{[X, Y]}, \quad X, Y \in \mathfrak{g}, \quad (3)$$

where $\{H_X, H_Y\}$ denotes the Poisson bracket generated by the symplectic structure ω . In usual terms the above assumption means the existence of a homomorphism $X \rightarrow H_X$ between \mathfrak{g} and the Lie algebra of functions on P .

A non-locally constant smooth function $I : \mathbb{R} \times P \rightarrow \mathbb{R}$ is called a *first integral* for (1) if it takes constant values on the integral curves of (1). If $I(t, x)$ is a periodic function on t will be called a *periodic first integral*. In terms of the

Poisson brackets on M , any first integral I satisfies

$$\frac{\partial I}{\partial t} + \sum_{i=1}^n b_i(t) \{H_i, I\} = 0.$$

To begin, we propose an ansatz for first integrals in the form

$$I(t, x) = \sum_{k=1}^n p_k(t) H_k(x) \quad (4)$$

for smooth functions $p_k(t)$, $k = 1, \dots, n$. In this case, functions $p_i(t)$ have to satisfy the linear system

$$\frac{\partial p_k}{\partial t} + \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij}^k b_i(t) p_j(t) = 0, \quad k = 1, \dots, n. \quad (5)$$

If we denote by $\mathcal{B} = \{e_1, \dots, e_n\}$ the basis for \mathfrak{g} corresponding to the vector fields $\{X_i \mid i = 1, \dots, n\}$, the system (5) becomes the Euler system on \mathfrak{g}

$$\frac{d\xi}{dt} = -[\phi(t), \xi], \quad \xi = \sum_{k=1}^n p_k e_k \quad (6)$$

where $\phi(t) = \sum_{i=1}^n b_i(t) e_i$ is a smooth curve on \mathfrak{g} .

Therefore, each solution of (6) gives us in (4) a first integral for (1). If we denote by $F(t) = ((F^{ij}(t)))$ the fundamental matrix of the linear system (6) with $F(0) = I$, any solution of (6) has the form $F(t)\alpha$ with $\alpha \in \mathfrak{g}$ and we have a basis of solutions given by

$$F(t)e_i = \sum_{j=1}^n F^{ij}(t)e_j, \quad i = 1, \dots, n$$

Moreover, if $\alpha = \sum_{j=1}^n \alpha_j e_j \in \mathfrak{g}$, the family of first integrals

$$I_\alpha(t, x) = \sum_{j=1}^n \sum_{k=1}^n \alpha_k F^{kj}(t) H_j(x), \quad \alpha \in \mathfrak{g} \quad (7)$$

generate a Lie algebra isomorphic to \mathfrak{g}

$$\{I_\alpha, I_\beta\} = I_{[\alpha, \beta]}, \quad \alpha, \beta \in \mathfrak{g}$$

Note that each element β of the center \mathfrak{z} of \mathfrak{g} is a singular point of the Euler system (6) and gives place to the time independent first integral

$$I_\beta(x) = \sum_{j=1}^n \beta_j H_j(x), \quad \beta = \sum_{k=1}^n \beta_k e_k \in \mathfrak{z}.$$

The space of first integrals

$$\{I_\beta \text{ where } \beta \in \mathfrak{z}\}$$

is an abelian Lie algebra of first integrals of (1).

Theorem 1 *The time dependent Hamiltonian system of Lie type (1), under the assumptions (2),(3), possesses a Lie algebra of first integrals isomorphic to its associated the Lie algebra \mathfrak{g} . If the center \mathfrak{z} of \mathfrak{g} is non trivial, the system has an abelian Lie algebra of time independent first integrals isomorphic to the center \mathfrak{z} .*

Remark 2 *If the Lie algebra \mathfrak{g} associated to the Hamiltonian system of Lie type (1) has no trivial center, we can take for the quotient Lie algebra $\mathfrak{g}/\mathfrak{z}$ a basis $\mathcal{D} = \{[K_1], [K_2], \dots, [K_k]\}$ where $k = \dim(\mathfrak{g}/\mathfrak{z})$ and $K_i = K_i(x)$ $i = 1, \dots, k$ are some fixed Hamiltonian functions representants of its class. Then each of the initial Hamiltonian vector fields $X_j(x)$, $j = 1, \dots, n$ can be written in the form $X_j = \sum_{i=1}^k c_{ji} Y_i(x) + Z_j(x)$, $j = 1, \dots, n$ where c_{ji} are scalars, $Y_i(x)$ for $i = 1, \dots, k$ is the Hamiltonian vector fields with Hamiltonian function K_i and $Z_i(x)$ are Hamiltonian vector fields belonging to the center \mathfrak{z} . To the initial time dependent Hamiltonia vector field (1) we can associate the Hamiltonian vector field of Lie type*

$$Y = \sum_{i=1}^k \sum_{j=1}^n b_j(t) c_{ji} Y_i(x)$$

generated by the Hamiltonian vector fields $Y_i(x)$ which close under the Lie bracket into the Lie algebra $\mathfrak{g}/\mathfrak{z}$. Then we have $X = Y + Z$ with $[Y, Z] = 0$ and each first integral $J_\mu(t, x)$ of Y having the form

$$J_\mu(t, x) = \sum_{s=1}^k q_s(t) K_s(x)$$

is also a first integral for (1). The previous discussion allows us to reduce the search of first integrals of the form (4) to the case of Lie algebras with trivial center.

3 Floquet theory for periodic Euler systems.

The Euler system (6) with $\phi(t)$ a T -periodic curve on \mathfrak{g} , is a periodic linear system on \mathfrak{g} . The fundamental matrix $F(t)$ with $F(0) = I$ preserves the Lie algebra structure

$$[F(t)x, F(t)y] = F(t)[x, y], \quad \forall x, y \in \mathfrak{g} \quad (8)$$

and

$$F(t+T) = F(t) \circ M$$

where $M = F(T)$ is the monodromy matrix. To the Euler system on \mathfrak{g} (6) we associate the T -periodic Lie system

$$\frac{dA}{dt} = -ad_{\phi(t)} \circ A, \quad A \in Ad(\mathfrak{g}) \quad (9)$$

on the adjoint group $Ad(\mathfrak{g})$ generated by the linear operators e^{ad_α} where $\alpha \in \mathfrak{g}$ and $ad_\alpha(\beta) = [\alpha, \beta]$, $\beta \in \mathfrak{g}$. The Lie system (9) possesses as fundamental solution the curve $F(t) \in Ad(\mathfrak{g})$. If the center \mathfrak{z} of the algebra is non trivial, each of its elements are proper vectors of each element of the matrix Lie group $Ad(\mathfrak{g})$ with proper value 1. In this case, for each $\alpha \in \mathfrak{z}$ the curve $F(t)\alpha$ is a periodic solution of (6) and we also have in (7) a periodic first integral for (1).

To prove the existence of periodic first integrals for (1) when \mathfrak{g} has trivial center, we consider Ad -invariant symmetric bilinear form on \mathfrak{g} given by the Killing form

$$\begin{aligned} \langle \cdot, \cdot \rangle_K : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathbb{R} \\ \langle x, y \rangle_K &= -tr(ad_x \circ ad_y), \quad x, y \in \mathfrak{g} \end{aligned}$$

The Ad -invariance of $\langle \cdot, \cdot \rangle_K$ takes the form

$$\langle [\alpha, \beta], \gamma \rangle_K + \langle \beta, [\alpha, \gamma] \rangle_K = 0, \quad \forall \alpha, \beta, \gamma \in \mathfrak{g} \quad (10)$$

From (10), the fundamental matrix $F(t)$ of (6) preserves the bilinear form

$$\langle F(t)\alpha, F(t)\beta \rangle_K = \langle \alpha, \beta \rangle_K, \quad \forall \alpha, \beta \in \mathfrak{g}$$

and particularly its monodromy operator satisfies

$$\langle M\alpha, M\beta \rangle_K = \langle \alpha, \beta \rangle_K, \quad \forall \alpha, \beta \in \mathfrak{g}.$$

Consider the complexification of the algebra $\mathfrak{g}^\mathbb{C}$ and the natural extension of the monodromy operator M to $\mathfrak{g}^\mathbb{C}$. If λ is a proper value of M with proper vector $\alpha \in \mathfrak{g}^\mathbb{C}$, we have

$$\langle M\alpha, M\bar{\alpha} \rangle_K = \lambda \bar{\lambda} \langle \alpha, \bar{\alpha} \rangle_K = \langle \alpha, \bar{\alpha} \rangle_K$$

and $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$ if $\langle \alpha, \bar{\alpha} \rangle_K \neq 0$. A proper vector α of M will be called *admissible* if $\langle \alpha, \bar{\alpha} \rangle_K \neq 0$. The possible proper values associated to admissible proper vectors are ± 1 or $e^{i\theta}$ with $\theta \neq 2\pi k$, $\forall k \in \mathbb{Z}$. If $\lambda = 1$, the solution $F(t)(\frac{\alpha + \bar{\alpha}}{2})$ is a periodic real solution; if $\lambda = -1$, the solution $F(t)(\frac{\alpha + \bar{\alpha}}{2})$ is anti-periodic real solution $F(t+T)\alpha = -F(t)\alpha$. If $\lambda = e^{i\theta}$ with $\theta \neq 2\pi k$, $\forall k \in \mathbb{Z}$, we can take the real vector

$$\delta = \frac{1}{2i}[\alpha, \bar{\alpha}]$$

and check using (8) that $M\delta = \delta$, and we have that $F(t)\delta$ is a T -periodic real solution of (6).

One can single out the following cases in which \mathfrak{g} admits a Ad -invariant bilinear and non-degenerate forms and consequently all proper vectors of M are admissible. (See [6])

a) The Lie algebra is semisimple,

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

and \langle, \rangle is defined as the Killing form

$$\langle x, y \rangle_{\mathfrak{g}} = -\text{tr}(ad_x \circ ad_y), \quad x, y \in \mathfrak{g}$$

b) The Lie algebra is compact, ie. the Killing form is negative semi-definite and its kernel is equal to the center of \mathfrak{g} , then there exists an Ad -invariant inner product on \mathfrak{g} .

We summarize the above discussion with the following

Theorem 3 *The T -periodic Hamiltonian system (1) possesses an abelian Poisson algebra of $2T$ -periodic first integrals. Moreover, if the Lie algebra \mathfrak{g} admits an Ad -invariant bilinear form, the system (1) has an abelian Poisson algebra of T -periodic first integrals.*

4 Application: The periodic Milne-Pinney equation.

The second order nonlinear differential equation

$$\frac{d^2 y}{dt^2} + w(t)^2 \frac{dy}{dt} + cy^{-3} = 0 \quad (11)$$

describes the time evolution of an oscillator with inverse quadratic potential and shares with the harmonic one the property of having a period independent of the energy [7]. The equation (11) obeys a superposition rule and first integrals have been obtained in [10]. Here, we apply our previous results and show the existence of a $sp(1, R)$ algebra of first integrals for equation (11). Moreover, we consider the periodic case in give a periodic first integral in terms of the periodic solution of the a periodic Euler equation on $sp(1, R)$.

Consider the symplectic manifold (M, Ω) , where $M = T^*\mathbb{R}^+$ with global coordinates (q, p) , $q > 0$ and symplectic form $\Omega = dq \wedge dp$. The periodic Milne-Pinney system is given by

$$\begin{aligned} \frac{dq}{dt} &= p \\ \frac{dp}{dt} &= -\omega^2(t)q + \frac{c}{q^3} \end{aligned} \quad (12)$$

where $c > 0$ and $\omega(t + 2\pi) = \omega(t)$.

The system (12) is a 2π -periodic Hamiltonian system with Hamiltonian function

$$H(t, q, p) = \frac{1}{2}p^2 + \frac{1}{2}(\omega^2(t)q^2 + \frac{c}{q^2}) \quad (13)$$

The Hamiltonian vector field

$$X = p \frac{\partial}{\partial q} - (\omega^2(t)q - \frac{c}{q^3}) \frac{\partial}{\partial p} \quad (14)$$

can be written in the form

$$X = -\alpha(t)X_2 - \beta(t)X_3$$

where $\alpha(t) = 1 - \omega^2(t)$ and $\beta(t) = \omega^2(t) + 1$. The vector fields X_2, X_3 are Hamiltonian vector fields

$$\begin{aligned} X_2 &= \frac{1}{2}(-p \frac{\partial}{\partial q} - (q + \frac{c}{q^3}) \frac{\partial}{\partial p}) \\ X_3 &= \frac{1}{2}(-p \frac{\partial}{\partial q} + (q - \frac{c}{q^3}) \frac{\partial}{\partial p}) \end{aligned}$$

with Hamiltonian functions H_2, H_3 given respectively by

$$\begin{aligned} H_2(q, p) &= -\frac{1}{4}p^2 + \frac{1}{4}(q^2 - \frac{c}{q^2}) \\ H_3(q, p) &= -\frac{1}{4}p^2 - \frac{1}{4}(q^2 + \frac{c}{q^2}) \end{aligned}$$

Taking the Lie bracket between X_2 and X_3 we denote by X_1 the Hamiltonian vector field given by

$$X_1 = [X_2, X_3] = \frac{1}{2}(q \frac{\partial}{\partial q} - p \frac{\partial}{\partial p})$$

with Hamiltonian function

$$H_1(q, p) = \frac{1}{2}pq$$

The commutation relations

$$[X_1, X_2] = -X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2 \quad (15)$$

correspond to those of the Lie algebra $sp(1, \mathbb{R})$. Then, the system (12) is a periodic Hamiltonian systems of Lie type. Moreover, the relation $\{H_X, H_Y\} = H_{[X, Y]}$ holds for any $X, Y \in sp(1, \mathbb{R})$.

Consider now, the Euler system on $sp(1, \mathbb{R})$ given by

$$\frac{d\xi}{dt} = [\alpha(t)e_2 + \beta(t)e_3, \xi], \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \quad (16)$$

where e_1, e_2, e_3 is the basis for $sp(1, \mathbb{R})$ with commuting relations given by (15). The system (16) takes the form

$$\begin{aligned}\frac{d\xi_1}{dt} &= -\beta(t)\xi_2 + \alpha(t)\xi_3 \\ \frac{d\xi_2}{dt} &= \beta(t)\xi_1 \\ \frac{d\xi_3}{dt} &= \alpha(t)\xi_1\end{aligned}$$

and can be written using the cross product in \mathbb{R}^3

$$-G \frac{d\xi}{dt} = \mu(t) \times \xi, \quad \xi \in \mathbb{R}^3 \quad (17)$$

where $\mu(t) = (0, \alpha(t), \beta(t))$ and $G = \text{diag}(1, 1, -1)$. Note that $K(\xi) = \xi_1^2 + \xi_2^2 - \xi_3^2$ is a first integral for (17). Moreover, taking into account that the periodic Euler system (17) possesses always periodic solutions [11], we have for the 2π -periodic Milne-Pinney equation a 2π -periodic first integral of the form

$$I(t, q, p) = \frac{1}{2}pq\xi_1(t) + (q^2 - p^2 - \frac{c}{q^2})\xi_2(t) - (q^2 + p^2 + \frac{c}{q^2})\xi_3(t)$$

where $\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ is a 2π -periodic solution for the Euler linear system (17).

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